MAY, 1884.

ANNALS OF MATHEMATICS.

EDITED BY

ORMOND STONE AND WILLIAM M. THORNTON.

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 1., Number 2.

CHARLOTTESVILLE, VA.:
Printed for the Editors by BLAKEY & PROUT, Steam Book and Job Printers.

Agents: B. WESTERMANN & CO., New York.

ANNALS OF MATHEMATICS.

Vol. I. May, 1884. No. 2.

ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

By Dr. G. W. Hill, Washington, D. C.

[continued from page 10.]

Also we get

$$2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} = -\left(2 + \frac{1}{12}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta) - \frac{3}{4}m\gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau)$$

$$\begin{aligned}
-6m^2(1 + \cos 2\tau)z\delta z = {} & 4\gamma\frac{\pi}{n}\cos(\zeta - \eta) + \left(2 - \frac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta - 2\tau) \\
& - 4\gamma\frac{\pi}{n}\cos(\zeta + \eta) + \left(2 - \frac{3}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta + 2\tau) \\
& - \left(2 - \frac{9}{4}m\right)\gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau) - 2\gamma\frac{\pi}{n}\cos(\zeta + \eta + 2\tau),
\end{aligned}$$

$$\begin{aligned}
6m^3\int z\delta z \sin 2\tau \,.\, d\zeta = {} & m\gamma\frac{\pi}{n}\cos(\zeta - \eta - 2\tau) + m\gamma\frac{\pi}{n}\cos(\zeta - \eta + 2\tau) \\
& + \gamma\frac{\pi}{n}\cos(\zeta + \eta - 2\tau),
\end{aligned}$$

and, by the addition of these three equations,

$$\begin{aligned}
U' = \gamma\frac{\pi}{n}\Big\{ & 2\cos(\zeta - \eta) + \left(2 + \frac{1}{4}m\right)\cos(\zeta - \eta - 2\tau) + \left(2 + \frac{1}{4}m\right)\cos(\zeta - \eta + 2\tau) \\
& - 4\cos(\zeta + \eta) - \left(1 - \frac{3}{2}m\right)\cos(\zeta + \eta - 2\tau) - 2\cos(\zeta + \eta + 2\tau)\Big\}.
\end{aligned}$$

In the next place

$$\begin{aligned}
2\frac{\pi}{n}x\frac{dz}{d\zeta} = \gamma\frac{\pi}{n}\Big\{ & \frac{3}{8}m\sin(\zeta - \eta + 2\tau) + \sin(\zeta + \eta) \\
& + \left(\frac{3}{8}m - \frac{21}{32}m^2\right)\sin(\zeta + \eta - 2\tau)\Big\},
\end{aligned}$$

$$3m^2 z\delta z \sin 2\tau = \gamma\frac{\pi}{n}\Big\{ (1 + \tfrac{3}{8}m) \sin(\zeta - \eta - 2\tau) - (1 + \tfrac{3}{8}m)\sin(\zeta - \eta + 2\tau)$$
$$- (1 + \tfrac{3}{8}m)\sin(\zeta + \eta - 2\tau) + \sin(\zeta + \eta + 2\tau) \Big\},$$

$$2\frac{d\lambda}{d\zeta} z\delta z = \gamma\frac{\pi}{n}\Big\{ \left(\frac{1}{2}m^{-1} - \frac{17}{48}m\right)\cos(\zeta - \eta - 2\tau)$$
$$+ \left(\frac{1}{2}m^{-1} - \frac{3}{16} - \frac{55}{384}m\right)\cos(\zeta - \eta + 2\tau)$$
$$+ \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{29}{9}\right)\cos(\zeta + \eta)$$
$$- (m^{-1} + \frac{103}{48} + \frac{8563}{1152}m)\cos(\zeta + \eta - 2\tau)$$
$$+ \frac{7}{3}\cos(\zeta + \eta + 2\tau) + \frac{3}{16}\cos(\zeta + \eta - 4\tau) \Big\}.$$

In these expressions the terms depending on the argument $\zeta - \eta$ are omitted because the co-efficient belonging to this argument in $\delta\lambda$ will be determined from the differential equation given specially for this purpose.

Remembering that

$$\frac{d\eta}{d\zeta} = 1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4,$$

the following expression for U'' is readily obtained: —

$$U'' = \gamma\frac{\pi}{n}\Big\{ \left(\frac{1}{2}m^{-1} + \frac{1}{2} + \frac{1}{3}m\right)\cos(\zeta - \eta - 2\tau)$$
$$+ \left(\frac{1}{2}m^{-1} + \frac{5}{16} + \frac{137}{384}m\right)\cos(\zeta - \eta + 2\tau)$$
$$+ \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{49}{18}\right)\cos(\zeta + \eta)$$
$$- \left(\frac{1}{2}m^{-1} + \frac{7}{3} + \frac{839}{144}m\right)\cos(\zeta + \eta - 2\tau)$$
$$+ \frac{25}{12}\cos(\zeta + \eta + 2\tau) + \frac{3}{16}\cos(\zeta + \eta - 4\tau) \Big\}.$$

Let us put now

$$r\delta r = \gamma\frac{\pi}{n}\Big\{ B_1\cos(\zeta - \eta) + B_2\cos(\zeta - \eta - 2\tau)$$
$$+ B_3\cos(\zeta - \eta + 2\tau) + B_4\cos(\zeta + \eta) + B_5\cos(\zeta + \eta - 2\tau)$$
$$+ B_6\cos(\zeta + \eta + 2\tau) \Big\},$$

$$\delta\lambda = \gamma\frac{\pi}{n}\Big\{ C_1 \sin(\zeta - \eta) + C_2 \sin(\zeta - \eta - 2\tau)$$
$$+ C_3 \sin(\zeta - \eta + 2\tau) + C_4 \sin(\zeta + \eta) + C_5 \sin(\zeta + \eta - 2\tau)$$
$$+ C_6 \sin(\zeta + \eta + 2\tau) + C_7 \cos(\zeta + \eta - 4\tau)\Big\}.$$

To a sufficient degree of approximation

$$C = 6m^2 \sin 2\tau,$$
$$D = -\frac{57}{8}m^4 - \frac{97}{4}m^5 + (3m^2 - m^4)\cos 2\tau,$$
$$E = 3m^2 \sin 2\tau.$$

Substituting the expressions for $r\delta r$ and $\delta\lambda$ in the differential equations which serve to determine them, the following equations of condition between the coefficients are obtained : —

$$B_1 = 2,$$
$$-(3 - 8m)B_2 + (3m^2 + \frac{3}{2}m^3)C_1 = 2 + \frac{1}{4}m,$$
$$-(3 - 8m)B_3 - (3m^2 + \frac{3}{2}m^3)C_1 = 2 + \frac{1}{4}m,$$
$$3B_4 = 4,$$
$$-B_5 - \left(\frac{3}{2}m^2 + \frac{9}{16}m^3\right)C_4 = 1 - \frac{3}{2}m,$$
$$15B_6 + 3m^2C_4 = 2,$$
$$(2 - 2m + \frac{1}{12}m^2)C_2 - \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)C_1 - 2B_2 = -\frac{1}{2}m^{-1} - \frac{1}{2} - \frac{1}{3}m,$$
$$(2 - 2m - \frac{17}{12}m^2)C_3 - \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)C_1 + 2B_3 = \frac{1}{2}m^{-1} + \frac{5}{16} + \frac{137}{384}m,$$
$$(2 + \frac{1}{12}m^2)C_4 + 2B_4 = \frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{49}{18},$$
$$\left\{\begin{aligned}&(2m + \frac{3}{4}m^2)C_5 - \left(\frac{3}{4}m + \frac{55}{32}m^2 + \frac{2417}{384}m^3\right)C_4\\ &-\frac{3}{4}mC_7 + 2B_5 + \frac{3}{4}mB_4\end{aligned}\right\} = -\frac{1}{2}m^{-1} - \frac{7}{3} - \frac{839}{144}m,$$
$$4C_6 - 2m^2C_4 + 2B_6 = \frac{25}{12},$$
$$2C_7 = -\frac{3}{16}.$$

To obtain an equation for determining C_1 we employ the special differential equation we have given for this purpose. Here we have

$$\frac{d\lambda^2}{d\zeta^2} = 1 + \frac{121}{32}m^4 + \left(\frac{11}{2}m^2 + \frac{85}{6}m^3\right)\cos 2\tau,$$

$$-\left(\frac{dr}{rd\zeta}\right)^2 = -2m^4,$$

$$\frac{21}{2}m^2(1+\cos 2\varphi) = \frac{21}{2}m^2 - \frac{231}{16}m^4 + \frac{21}{2}m^2\cos 2\tau,$$

$$\frac{d\lambda^2}{d\zeta^2} - \left(\frac{dr}{rd\zeta}\right)^2 + \frac{21}{2}m^2(1+\cos 2\varphi) = 1 + \frac{21}{2}m^2 - \frac{405}{32}m^4 + (16m^2 + \frac{85}{6}m^3)\cos 2\tau.$$

Retaining only the term whose argument is $\zeta - \eta$,

$$\left\{\frac{d\lambda^2}{d\zeta^2} - 1 - \left(\frac{dr}{rd\zeta}\right)^2 + \frac{21}{2}m^2(1+\cos 2\varphi)\right\} z\delta z$$
$$= -(7 - \frac{11}{8}m - \frac{1535}{192}m^2)\,\gamma\frac{\pi}{n}\cos(\zeta - \eta).$$

In addition

$$\frac{dr}{rd\zeta} = (2m^2 + \frac{13}{3}m^3)\sin 2\tau,$$

$$\frac{dr}{rd\zeta}\frac{d(z\delta z)}{d\zeta} = -(m + \frac{223}{48}m^2)\,\gamma\frac{\pi}{n}\cos(\zeta - \eta),$$

$$-4\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} = (4 + \frac{1}{6}m^2)\,\gamma\frac{\pi}{n}\cos(\zeta - \eta).$$

Let us write the series for z

$$z = \gamma\left\{q_1\sin\eta + q_2\sin(2\tau - \eta) + q_3\sin(2\tau + \eta)\right\},$$

then

$$z\delta z = \frac{1}{2}(A_1q_1 - A_2q_2 + A_3q_3)\,\gamma\frac{\pi}{n}\cos(\zeta - \eta),$$

$$\frac{dz}{d\zeta} = \gamma\left\{(1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4)q_1\cos\eta\right.$$
$$\left. + (1 - 2m - \frac{3}{4}m^2)q_2\cos(2\tau - \eta) + 3q_3\cos(2\tau + \eta)\right\},$$

$$\frac{d.\delta z}{d\zeta} = \frac{\pi}{n}\left\{A_1\cos\zeta - (1 - 2m)A_2\cos(\zeta - 2\tau) + 3A_3\cos(\zeta + 2\tau)\right\},$$

$$z\delta z - \frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta} = -\frac{1}{2}\left\{\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4\right)A_1q_1\right.$$
$$\left. + (4m - \frac{13}{4}m^2)A_2q_2 + 8A_3q_3\right\}\gamma\frac{\pi}{n}\cos(\zeta - \eta).$$

Substituting in the last equation the values of A_1, q_1, etc., it becomes

$$z\delta z - \frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta} = \left(\frac{1}{2} - \frac{3}{8}m - \frac{253}{96}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta - \eta).$$

Also we have

$$-3m^2(1+3\cos 2\tau)\,r\delta r = \left[3m^2B_1 + \frac{9}{2}m^2(B_2+B_3)\right]\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

but from the previous equations of condition, $B_1 = 2$, and $B_2 + B_3 = -\frac{4}{3}$, hence

$$-3m^2(1+3\cos 2\tau)\,r\delta r = 0.$$

In addition

$$\frac{dR}{d\lambda} = -\frac{3}{2}m^2\sin 2\tau,$$

$$-7\frac{dR}{d\lambda}\delta\lambda = -\frac{21}{4}m^2(C_2-C_3)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$r^2\frac{d\lambda}{d\zeta} = 1 - \frac{1}{3}m^2 + \left(\frac{3}{4}m^2 + \frac{3}{4}m^3\right)\cos 2\tau,$$

$$-3m\int\delta.\frac{dR}{d\lambda}d\zeta = 3m\int\left[D\delta\lambda + E(r\delta r - z\delta z)\right]d\zeta,$$

$$3m\int D\delta\lambda d\zeta = -\left\{\left(\frac{57}{2}m^3 + \frac{1723}{16}m^4\right)C_1 - (6m+\frac{9}{4}m^2)(C_2+C_3)\right\}\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$3m\int Er\delta r d\zeta = (6m+\frac{9}{4}m^2)(B_2-B_3)\gamma\frac{\pi}{n}\cos(\zeta-\eta),$$

$$-3m\int Ez\delta z d\zeta = -\left(\frac{9}{16}m - \frac{27}{64}m^2\right)\gamma\frac{\pi}{n}\cos(\zeta-\eta).$$

Thus is obtained the equation which determines C_1: —

$$\left\{\begin{array}{l}\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{305}{128}m^4\right)C_1 \\ -\frac{9}{2}m^2(C_2-C_3) - \left(\frac{57}{2}m^3 + \frac{1723}{16}m^4\right)C_1 \\ +(6m+\frac{9}{4}m^2)(B_2-B_3+C_2+C_3)\end{array}\right\} = \frac{5}{2} + \frac{9}{16}m - \frac{125}{96}m^2.$$

But the previous equations of condition furnish

$$C_2 - C_3 = -\frac{1}{2}m^{-1} - \frac{215}{96},$$

$$B_2 - B_3 + C_2 + C_3 = \left(\frac{19}{4}m^2 + \frac{97}{6}m^3\right)C_1 - \frac{3}{32} + \frac{27}{256}m,$$

consequently

$$\left(\frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{305}{128}m^4\right)C_1 = \frac{5}{2} - \frac{9}{8}m - \frac{1133}{96}m^2,$$

and

$$C_1 = \frac{10}{3}m^{-2} - \frac{1}{4}m^{-1} - \frac{503}{96}.$$

Solving the remaining equations of conditions we get

$$\begin{aligned}
\delta\lambda = & \left(\frac{10}{3}m^{-2} - \frac{1}{4}m^{-1} - \frac{503}{96}\right)\gamma\frac{\pi}{n}\sin(\zeta - \eta) \\
& - \left(\frac{1}{4}m^{-1} - \frac{41}{12}\right)\gamma\frac{\pi}{n}\sin(\zeta - \eta - 2\tau) \\
& + \left(\frac{1}{4}m^{-1} + \frac{181}{32}\right)\gamma\frac{\pi}{n}\sin(\zeta - \eta + 2\tau) \\
& + \left(\frac{2}{3}m^{-2} + \frac{1}{4}m^{-1} + 0\right)\gamma\frac{\pi}{n}\sin(\zeta + \eta) \\
& + \left(\frac{3}{2}m^{-1} - \frac{235}{96}\right)\gamma\frac{\pi}{n}\sin(\zeta + \eta - 2\tau) \\
& + \frac{41}{48}\gamma\frac{\pi}{n}\sin(\zeta + \eta + 2\tau) \\
& - \frac{3}{32}\gamma\frac{\pi}{n}\sin(\zeta + \eta - 4\tau).
\end{aligned}$$

The expression for the inequalities in latitude is

$$\begin{aligned}
\delta\beta = \frac{\delta z}{r} = & -\left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{11}{3} + \frac{3347}{288}m\right)\frac{\pi}{n}\sin\zeta \\
& + \left(\frac{1}{2}m^{-1} + \frac{17}{12} + \frac{1187}{288}m\right)\frac{\pi}{n}\sin(\zeta - 2\tau) \\
& - \left(\frac{11}{12} + \frac{1043}{288}m\right)\frac{\pi}{n}\sin(\zeta + 2\tau) \\
& + \frac{11}{32}m\frac{\pi}{n}\sin(\zeta - 4\tau) \\
& + \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{35}{48}\right)e\frac{\pi}{n}\sin(\zeta - \xi). \\
& - \left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{13}{3}\right)e\frac{\pi}{n}\sin(\zeta + \xi) \\
& + \left(2m^{-1} + 2\right)e\frac{\pi}{n}\sin(\zeta - 2\tau + \xi) \\
& - \left(\frac{5}{2}m^{-1} + \frac{527}{48}\right)e\frac{\pi}{n}\sin(\zeta + 2\tau - \xi) \\
& + \left(\frac{1}{2}m^{-1} + \frac{7}{24}\right)e\frac{\pi}{n}\sin(\zeta - 2\tau - \xi)
\end{aligned}$$

$$-\frac{7}{3}e^{\frac{\pi}{n}}\sin(\zeta + 2\tau + \xi)$$
$$+\frac{15}{16}e^{\frac{\pi}{n}}\sin(\zeta - 4\tau + \xi).$$

[TO BE CONTINUED.]

NOTE ON A UNIQUE PROPERTY OF AN AXISYMMETRIC DETERMINANT OF THE FOURTH ORDER.

By PROF. THOMAS MUIR, Bishopton, Scotland.

The determinant in question is

$$\begin{vmatrix} 0 & a & b & c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & k \end{vmatrix}, \text{ or } \Omega \text{ say,}$$

its only specialty as an axisymmetric determinant of the fourth order being that it has a zero in the place (1,1). Now if we take the minor

$$\begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix}, \text{ or } K \text{ say,}$$

and substitute in it for a and b the complementary minors of h and f in Ω, it is found that, curiously enough, the new determinant is divisible by the old, the quotient being

$$\begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}$$

That is to say, if H and F denote the complementary minors of h and f in Ω we assert the identity

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} 0 & a & b \\ a & d & e \\ b & e & g \end{vmatrix} \cdot \begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix}.$$

This may be established in various ways, but perhaps the following is as neat as any other : —

Multiplying the left-hand member column-wise by

$$\begin{vmatrix} 1 & 0 & 0 \\ bc & 1 & 0 \\ -ac & 0 & 1 \end{vmatrix}$$

and bearing in mind that

$$H = \begin{vmatrix} 0 & a & b \\ a & d & e \\ c & f & h \end{vmatrix} = abf + ace - bcd - a^2h,$$

and

$$F = \begin{vmatrix} 0 & a & b \\ b & e & g \\ c & f & h \end{vmatrix} = b^2f + acg - bce - abh,$$

we have

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} bcH - acF & a(bf-ah) & b(bf-ah) \\ H & d & e \\ F & e & g \end{vmatrix}.$$

Multiplying again by the same multiplier, but now performing the operation row-wise in order that the determinant may re-assume the original symmetrical form, we have

$$\begin{vmatrix} 0 & H & F \\ H & d & e \\ F & e & g \end{vmatrix} = \begin{vmatrix} bcH - acF & a(bf-ah) & b(bf-ah) \\ a(bf-ah) & d & e \\ b(bf-ah) & e & g \end{vmatrix}.$$

But since

$$\begin{vmatrix} 0 & a & b & 0 \\ a & d & e & a \\ b & e & g & b \\ c & f & h & c \end{vmatrix} = 0,$$

it follows that

$$cK - bH + aF = 0,$$

and thus

$$bcH - acF = c^2K.$$

Substituting this in our last obtained result, and partitioning the determinant into two, we have

$$\begin{vmatrix} c^2K & 0 & 0 \\ 0 & d & e \\ 0 & e & g \end{vmatrix} + \begin{vmatrix} 0 & a(bf-ah) & b(bf-ah) \\ a(bf-ah) & d & e \\ b(bf-ah) & e & g \end{vmatrix},$$

i. e.,

$$c^2K(dg-e^2)+(bf-ah)^2K,$$

i. e.,

$$K[c^2(dg-e^2)+(bf-ah)^2],$$

i. e.,

$$K\begin{vmatrix} 0 & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & 0 \end{vmatrix},$$

as was to be proved.

In connection with the co-factor of K here it is worth while remarking upon the very considerable cancelling effect of altering the sign of one of the non-axial elements in an axisymmetric determinant. If the determinant be of the n^{th} order the number of terms is of course $n!$, or classifying them as they are got when we expand the determinant according to binary products of elements from the first row and first column, we may say that the number of terms is

$$(n-1)!+(n-2)!\,(n-1)+2C_{n-1,2}\,(n-2)!.$$

Now if one of the non-axial elements be altered in sign, say the element in the place $(1, n)$, the number of terms obtained in like manner is

$$(n-1)!+(n-2)!\,(n-1)+2C_{n-2,2}\,(n-2)!.$$

The number in the latter case is thus diminished by

$$2\,(n-2)!\,(C_{n-1,2}-C_{n-2,2})$$

i. e.,

$$2\,(n-2)!\,(n-2).$$

ON THE STRENGTH OF TELEGRAPH WIRES.

By Prof. Wm. M. Thornton, University of Virginia.

1. The use of the flexible cable or wire as a constructive element suggests a variety of problems possessing more or less interest both for the geometer and for the engineer. The following note is an attempt to treat those problems which arise in the comparatively little studied case of the suspended wires used for telegraph lines, telephone lines, and the like.

2. We assume that the wire is perfectly flexible, homogeneous, and of uniform diameter. These assumptions are so near to the truth that the results deduced from them may be applied to the solution of constructive problems. The forces which act on the wire are its weight and the pressure of the wind blowing against it. The former is estimated at 480 pounds to the cubic foot; the latter at 75 per cent. of the pressure on the elevation of the wire. The joint load therefore in the most unfavorable case, where the two forces coincide in direction, will be in tons per inch of length

$$q = \frac{79W + 222PW^{\frac{1}{2}}}{10^{10}}$$

where W is the weight of the wire in pounds per mile of length, and P the greatest wind pressure in pounds per square foot.

The greatest tension admissible in such a wire is in tons

$$T = \frac{\sigma W}{17600}$$

where σ is the modulus of tenacity in tons per square inch.

3. The general differential equation of the catenary of such a wire is

$$dy = \frac{Q}{H} \cdot dx$$

where Q denotes the total load applied to the arc measured from the lowest point to (x, y) and H the tension in the wire at that lowest point. Two cases must be distinguished. In the first, the catenary is so flat that the load carried by the wire may be treated as uniformly distributed over the span. In the second, the curvature is sharper, and the load is treated as uniformly distributed over the arc.

I. LOAD UNIFORM PER INCH OF SPAN.

4. In this case $Q = qx$, where q is constant and the catenary

$$y = \frac{qx^2}{2H}$$

is the common parabola. The length of the arc may be expressed in finite terms by elementary transcendents. For actual computations, however, it is more convenient to develop dl into a series, thus: —

$$\frac{dl}{dx} = 1 + \frac{1}{2}\left(\frac{qx}{H}\right)^2 - \frac{1}{8}\left(\frac{qx}{H}\right)^4 + \frac{1}{16}\left(\frac{qx}{H}\right)^6 - \dots;$$

whence for the length l of the arc measured from the lowest point (0,0) to (x, y), we get if $y = \rho x$

$$\frac{l}{x} = 1 + \frac{2}{3}\rho^2 - \frac{2}{5}\rho^4 + \frac{4}{7}\rho^6 - \dots.$$

In all problems in which ρ^4 is negligible this approximate formula furnishes results practically coincident with those of the more exact formula given below. The principal questions which arise in practice are as follows: —

5. We put s for span, f for dip of wire, and ρ for the ratio $2f : s$. Then the horizontal tension H in the wire and the sloping tension T are respectively

$$H = \frac{qs}{4\rho}, \qquad T = \sqrt{\left\{H^2 + \left(\frac{qs}{2}\right)^2\right\}}.$$

The following sub-cases are noted: —

1′. Given span, dip, wind-pressure, weight, and strength of wire; to find the stress therein.

$$\sigma = 8800\, s \sqrt{\left(1 + \frac{1}{4\rho^2}\right)} \cdot \frac{q}{W}.$$

$$\frac{q}{W} = \frac{79 + 222\, PW^{-\frac{1}{2}}}{10^{10}}.$$

2′. Given span, wind-pressure, weight, and strength of wire; to find the dip.

$$\frac{1}{4\rho^2} = \left(\frac{2}{s} \cdot \frac{T}{q}\right)^2 - 1,$$

$$\frac{T}{q} = \frac{\sigma}{17600} \cdot \frac{q}{W},$$

and $\frac{q}{W}$ is found as above.

3′. Given dip, wind-pressure, weight, and strength of wire; to find the span.

$$\frac{1}{\rho^2} = \sqrt{\left\{1 + \left(\frac{2}{f} \cdot \frac{T}{q}\right)^2\right\}} - 1,$$

and $\frac{T}{q}$ is found as above.

4′. Given span, dip, wind-pressure, and strength of wire; to find its weight.

$$\therefore W = \frac{222\,P}{\dfrac{10^{10}}{\left(\dfrac{W}{q}\right)} - 79},$$

$$\frac{W}{q} = \frac{8800\,s}{\sigma}\sqrt{\left\{1 + \frac{1}{4\rho^2}\right\}}.$$

Thus, for example, in a wire admitting a working stress of 12 tons on the square inch, if the span is 200′, the dip 30′, and the greatest anticipated wind-pressure 30 pounds on the square foot,

$$\rho = \frac{1}{40},$$

$$\frac{W}{q} = 1760000\sqrt{401},$$

$$\frac{10^{10}}{\left(\dfrac{W}{q}\right)} = 283.7,$$

$$\therefore W = 31.76,$$

and the necessary weight of wire for the line is 1009 pounds to the mile

II. LOAD UNIFORM PER INCH OF ARC.

6. In this case $Q = qs$ and the differential equation to the catenary is

$$dy = \frac{q}{H}s\,dx.$$

This is the familiar case of the common catenary. The equation in integral form is, with the origin at the lowest point,

$$y = \frac{H}{2q}\left(e^{\frac{qx}{H}} + e^{-\frac{qx}{H}}\right) - \frac{H}{q},$$

and the length of the arc from this origin to (x, y) is

$$l = \frac{H}{2q}\left(e^{\frac{qx}{H}} - e^{-\frac{qx}{H}}\right).$$

These exponential forms are for more convenient use in practical applications developed into infinite series. The following are the results which furnish values of x, y, and l in terms of $\rho = y : x$:—

$$x = \frac{2H}{q}\left(\rho - \frac{1}{3}\rho^3 + \frac{13}{45}\rho^5 - \ldots\right),$$

$$y = \frac{2H}{q}(\rho^2 - \frac{1}{3}\rho^4 + \frac{13}{45}\rho^6 - \ldots),$$

$$\frac{l}{x} = 1 + \frac{2}{3}\rho^2 - \frac{14}{45}\rho^4 + \frac{278}{945}\rho^6 - \ldots .$$

These more exact formulæ should be used in all cases in which the fourth or higher powers of ρ cannot be neglected; as, for example, in designing great river spans and the like. The reduction of the results to forms suitable for computation will now be given. The series have been found by actual trial to furnish results with greater accuracy and facility than the tables of hyperbolic sines and cosines.

7. We put as before s for span, f for dip, and ρ for the ratio $2f:s$. Then

$$T = H + qf,$$

and

$$H = \frac{qs}{4\rho}(1 + \frac{1}{3}\rho^2 - \frac{8}{45}\rho^4 + \ldots).$$

The principal sub-cases are noted.

1′. Given span, dip, wind-pressure, and weight of wire; to find the stress therein.

$$\sigma = \frac{17600}{W} \cdot \frac{T}{q} \cdot q,$$

$$\frac{T}{q} = \frac{H}{q} + f,$$

$$\frac{H}{q} = \frac{s}{4\rho}(1 + \frac{1}{3}\rho^2 - \frac{8}{45}\rho^4 + \ldots),$$

and q is found from formula in Art. 2.

2′. Given dip, wind-pressure, weight, and strength of wire; to find the span. We compute q and T from formulæ in Art. 2; then

$$H = T - qf,$$

and s is found from the transcendental equation

$$e^{\frac{qs}{2H}} = \sqrt{\frac{T+H}{2H}} + \sqrt{\frac{T-H}{2H}}$$

by the aid of a table of Naperian logarithms.

3′. Given span, wind-pressure, weight and strength of wire; to find the dip. We compute q and T from formulæ in Art. 2, and $\frac{q}{H}$ from the transcendental equation

$$\frac{qs}{2H} = \frac{qs}{4T}\left(e^{\frac{qs}{2H}} + e^{-\frac{qs}{2H}}\right) = \frac{qs}{2T}\operatorname{coh}\frac{qs}{2H}.$$

Then

$$f = \frac{T}{q} - \frac{H}{q}.$$

As the solution of such equations by trial and error is a tedious process, we express $\frac{qs}{2H} = h$ in terms of $\frac{qs}{2T} = t$ as follows: —

By developing the exponentials into series and dividing we find

$$t = h - \frac{1}{2}h^3 + \frac{5}{24}h^5 - \frac{61}{720}h^7 + \ldots.$$

Reverting the series we get

$$h = t + \frac{1}{2}t^3 + \frac{13}{24}t^5 + \frac{541}{720}t^7 + \ldots.$$

Having computed t from the original data we find h from this series and then calculate

$$f = \frac{s}{2t} - \frac{s}{2h}.$$

4'. Given span, dip, wind-pressure, and strength of wire; to find its weight. It is easy to show that

$$\frac{T}{q} = \frac{s}{4\rho}\left(1 + \frac{7}{3}\rho^2 - \frac{8}{45}\rho^4 + \ldots\right),$$

whence

$$\frac{W}{q} = \frac{17600}{\sigma}\cdot\frac{T}{q}$$

and

$$\sqrt{W} = \frac{222P}{\dfrac{10^{10}}{\left(\dfrac{W}{q}\right)} - 79}.$$

For example, a river crossing of $2000'$ constructed of steel wire weighing 550 pounds to the mile, and possessing a working strength of 30 tons to the square inch, if exposed to a wind pressure of 40 pounds to the square foot should have a dip computed as follows. We find

$$q = 0.000025\,17,$$
$$T = 0.9375;$$

$$\therefore t = 0.3222,$$
$$h = 0.3411\,;$$

whence

$$f = \frac{1000}{t} - \frac{1000}{h} = 172^f.$$

8. It is sufficiently obvious that the dip corresponding to the greatest admissible stress is that which the wire must have when the temperature is at the lowest. The line will ordinarily be erected at a higher temperature and the necessary increment must be given to the dip. To determine this increment we compute the length of wire l in the whole span from the formula

$$\frac{l}{s} = 1 + \frac{2}{3}\rho^2 - \frac{14}{45}\rho^4 + \frac{278}{945}\rho^6 - \ldots,$$

and then calculate the increased length L at the higher temperature, using the appropriate co-efficient of dilation; which for iron or steel wire is about 1:150,000 for each degree Fahrenheit. Then from the same formula we compute the value P of ρ corresponding to the increased value L of l, and thence the increased value F of the dip. For convenience we revert this series also. Putting λ for $\frac{l}{s} - 1$ we find

$$\rho^2 = \frac{3}{2}\lambda + \frac{21}{20}\lambda^2 - \frac{27}{1400}\lambda^3 + \ldots.$$

For example, in the case just treated, if the lowest temperature be $-20°$ F. and the temperature on the day of erection be $55°$ F., we find

$$\rho = 0.1720\,; \qquad \therefore \frac{l}{s} = 1.0194.$$

Increasing this by 1 : 20,000 we get

$$\frac{L}{s} = 1.0199\,;$$

$$\therefore \lambda = 0.0199, \qquad \rho^2 = 0.0303, \qquad \rho = 0.174,$$

and the increased dip is 174^f, the length of cable at the same time being 2040^f, An apparent increase of precision in such results can be readily attained by the more liberal use of decimal places. But the uncertainty of the data, particularly of the wind-pressure, is so great that no confidence should be bestowed on the results of such calculations. It may be remarked that in general the data in engigineering problems can themselves be relied on only to three significant figures and the results of computations based on them can of course possess no higher degree of precision.

A PROJECTIVE RELATION AMONG INFINITESIMAL ELEMENTS.

By PROF. J. E. OLIVER, Ithaca, N. Y.

THEOREM.—If s_1, s_2, s_3 be the lengths of the finitely-distant infinitesimal collinear elements A_1B_1, A_2B_2, A_3B_3; and if s'_1, s'_2, s'_3 be their projections upon another right line by a pencil with any vertex; and if l_1, l_2, l_3 be certain constants, viz: $l_1 = A_1B_1 \,.\, \overline{A_2B_3}^2$, $l_2 =$ etc.; and if the signs of the radicals be rightly taken, then

$$(1) \qquad \left(\frac{l_1}{s'_1}\right)^{\frac{1}{2}} + \left(\frac{l_2}{s'_2}\right)^{\frac{1}{2}} + \left(\frac{l_3}{s'_3}\right)^{\frac{1}{2}} = 0.$$

So, if $t_1, \ldots t_4$ and $t'_1, \ldots t'_4$, be the areas of the finitely-distant infinitesimal complanar triangles $A_1B_1C_1, \ldots A_4B_4C_4$ and of the projections, and if their constants $m_1 = t_1 \,.\, (\text{area } \overline{A_2A_3A_4})^3$, $m_2 =$ etc., then

$$(2) \qquad \left(\frac{m_1}{t'_1}\right)^{\frac{1}{3}} + \ldots + \left(\frac{m_4}{t'_4}\right)^{\frac{1}{3}} = 0.$$

So, if $v_1, \ldots v_5$ and $v'_1, \ldots v'_5$ be the volumes of tetrahedrons and of their projections, then

$$(3) \qquad \left(\frac{n_1}{v'_1}\right)^{\frac{1}{4}} + \ldots + \left(\frac{n_5}{v'_5}\right)^{\frac{1}{4}} = 0,$$

where the constants $n_1 = v_1 \,.\, (\text{vol. } \overline{A_2A_3A_4A_5})^4$, $n_2 =$ etc.

And similarly for "homaloidal space" of 4, 5, or more dimensions.

The triangular, tetrahedral, etc., elements in (2), (3), etc., are taken infinitesimal in all their dimensions, and are evidently replaceable by any other elements that are so; e. g., by infinitesimal circles and spheres deforming into ellipses and ellipsoids.

PROOF OF (1). The anharmonic ratio

$$\frac{-A'_2B'_3 \,.\, B'_2A'_3}{A'_2B'_2 \,.\, A'_3B'_3} = \frac{-A_2B_3 \,.\, B_2A_3}{A_2B_2 \,.\, A_3B_3};$$

or in the limit, when points A coincide with points B,

$$\overline{A'_2A'_3}^2 = \frac{s'_2\, s'_3}{s_2\, s_3} \,.\, \overline{A_2A_3}^2 = \frac{s'_1 s'_2 s'_3}{s_1 s_2 s_3} \,.\, \frac{l_1}{s'_1}.$$

So,

$$\overline{A'_3A'_1}^2 \text{ and } \overline{A'_1A'_2}^2 = \text{etc., by symmetry.}$$

But

$$\overline{A'_2A'_3} + \overline{A'_3A'_1} + \overline{A'_1A'_2} = 0,$$

wherein, as in Tait's Quaternions, the vinculums show that the segments are measured in the directions indicated.

Hence the formula (1).

The proofs of (2), (3), etc., are similar; e. g., it is known that the ratio of products of triangle-areas

$$\frac{\overline{A'_2B'_3C'_4} \cdot \overline{B'_2C'_3A'_4} \cdot \overline{C'_2A'_3B'_4}}{\overline{A'_2B'_2C'_2} \cdot \overline{A'_3B'_3C'_3} \cdot \overline{A'_4B'_4C'_4}} = \frac{\overline{A_2B_3C_4} \cdot \overline{B_2C_3A_4} \cdot \overline{C_2A_3B_4}}{\overline{A_2B_2C_2} \cdot \overline{A_3B_3C_3} \cdot \overline{A_4B_4C_4}},$$

because each member is a function of only the angles made at the centre of projection by the different projecting rays; hence, in the limit,

$$\overline{A'_2A'_3A'_4}^3 = \frac{t'_1 \ldots t'_4}{t_1 \ldots t_4} \cdot \frac{m_1^3}{t'_1}, \text{ etc.},$$

where the vinculum shows that the area is taken positive or negative according as the cyclic order of the vertices corresponds to positive or negative rotation. But

$$\overline{A'_2A'_3A'_4} + \overline{A'_1A'_4A'_3} + \overline{A'_4A'_1A'_2} + \overline{A'_3A'_2A'_1} = 0,$$

which gives (2).

The rule of signs for linear, triangular, pyramidal, and other elements is like that used in determinants, viz: every interchange of two letters under the vinculum changes the value of the symbol from positive to negative or the reverse. When no vinculum is used, the length or area, etc. indicated is taken positive, or rather signless.

Since the location of the centre of projection, and of $A'_1B'_1 \ldots A'_2 \ldots$, always requires one dimension beside those occupied by $A_1B_1 \ldots A_2 \ldots$, the proof and even the statement of (3) is 4-dimensional. We might avoid this by defining $A'_1B'_1 \ldots D'_5$ as any figure homographic to $A_1B_1 \ldots D_5$, and by using this principle, that if in two homographic figures coresponding tetrahedral volumes be each divided by the product of its vertices' distances from the centre of homology, then the quotients have a fixed ratio. A like treatment would apply to (2), etc. Indeed, homographic figures are only projected figures so moved without further distortion as to be collinear, complanar, or co-spatial, etc. But projection, in however many dimensions, appears to me the more natural procedure.

A NEW DESCRIPTION OF CONICS.

By Prof. J. W. Nicholson, Baton Rouge, La.

The locus of the first order, or right-line, is represented by the equation

$$y = ax + b.$$

This may have, with reference to the x-axis any one of three directions: it may be parallel to it, in which case $a = 0$; it may make an acute angle with it, in which case $a > 0$; it may make an obtuse angle with it, in which case $a < 0$. In the first case y is constant; in the second y is an increasing linear function of x; in the third case y is a decreasing linear function of x. The equation may therefore be written under one of the following forms:

$$y = C, \qquad y = Y, \qquad y = \lambda,$$

where C denotes a constant, Y an increasing linear function of x, λ a decreasing linear function of x.

The locus of the second order, or conic section, is represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which by appropriate transformations is reducible to the form

$$y^2 = Px^2 + Qx + R,$$

and represents in general a parabola, an ellipse, or an hyperbola as

$$P = 0 < > 0.$$

Observing that the second member of the reduced equation can always be identified with the product

$$(a_1x + b_1)(a_2x + b_2) = a_1a_2x^2 + (a_1b_2 + a_2b_1)x + b_1b_2,$$

and that conversely it is in general resolvable into two real linear factors of the form $a_1x + b_1$, $a_2x + b_2$ we are justified in asserting the following propositions:—

I. The locus whose ordinate is a mean proportional between the corresponding ordinates of two given right lines is a conic.

II. Conversely, every conic can be generated by a point which moves so that its ordinate is a mean proportional between the corresponding ordinates of two suitably located right lines.

From these propositions the following corollaries flow:—

1. If a_1 or $a_2 = 0$, $P = 0$ and the conic is a parabola; conversely, every parabola can be represented by an equation of the form

$$y^2 = CY \text{ or } C\lambda.$$

2. If a_1, a_2 are of opposite signs, $P < 0$ and the conic is an ellipse; conversely, every ellipse can be represented by an equation of the form

$$y^2 = YX.$$

3. If a_1, a_2 are of like signs, $P > 0$ and the conic is an hyperbola; conversely, every hyperbola can be represented by an equation of the form

$$y^2 = Y_1 Y_2.$$

These propositions furnish easy tests of the nature of any given locus of the second order. The following illustrative examples are added: —

a. To determine the figure of the section of a right circular cone by a plane.

VFG is the cone; AHK the plane; FQR a parallel circle intersecting the plane in P. Cut by a plane through V normal to the secant plane, intersecting the latter in AJ, the base in FG, and the parallel in QR, and join PM, M being the intersection of AJ and QR. Then PM is the ordinate of the required curve and

$$PM^2 = QM \,.\, MR.$$

Now MR, the ordinate of VG relative to AJ, is increasing; hence the locus is an ellipse, a parabola, or an hyperbola, as QM is decreasing, constant, or increasing; that is, as the angle GAJ is greater than, equal to, or less than the vertical angle of the cone.

b. To find the locus of a point whose distance from a fixed circle measured on a radius equals its distance from a fixed diameter of that circle measured on on a line of fixed direction.

Take the centre for origin and the fixed diameter for y-axis, and let the fixed direction make with the x-axis the angle a. Then the equation to the locus is

$$\begin{aligned} y^2 &= (r + x \sec a)^2 - x^2 \\ &= [r + x(\sec a + 1)] \,.\, [r + x(\sec a - 1)], \end{aligned}$$

and represents a parabola or an hyperbola as $\sec a = > 1$.

[Professor Nicholson observes in a letter to the editors that the factors of $Px^2 + Qx + R$ will, if the axes are suitably chosen, be real when the conic is real, and imaginary when the conic is imaginary. The criteria of form in the latter case appear to fail.—Ed.]

SOLUTIONS OF EXERCISES.

1

A horizontal wind blows against a hemispherical dome of radius R^f. The pressure exerted by the wind on a plane surface normal to its direction is P pounds to the square foot; on a surface oblique to its direction the pressure exerted is normal, but is reduced in the ratio (Poncelet, *Mécanique Industrielle*, 403)

$$1 : 1 + \frac{1}{2}\tan^2 i,$$

where i is the angle of incidence. It is required to find the magnitudes and the points of application of the horizontal and vertical components of the resultant wind-pressure. [*W. M. Thornton.*]

SOLUTION.

Take the axis of x in the direction of the wind, and the plane of xy for the horizontal plane. The element of the surface is

$$dA = \sqrt{1 + p^2 + q^2} \,.\, dx\, dy;$$

and since the equation of the surface is $x^2 + y^2 + z^2 = R^2$,

$$p = \frac{dz}{dx} = -\frac{x}{z}, \qquad q = \frac{dz}{dy} = -\frac{y}{z}, \qquad dA = \frac{R}{z} \,.\, dx\, dy.$$

We have also $\cos i = \dfrac{x}{R}$, and therefore

$$1 + \frac{1}{2}\tan^2 i = \frac{R^2 + x^2}{2x^2}.$$

The normal pressure on the element of surface is

$$\frac{2PRx^2}{z\,(R^2 + x^2)} \,.\, dx\, dy.$$

Put $a^2 = R^2 - x^2$, so that $z = \sqrt{a^2 - y^2}$; and since we have to multiply the normal pressure by $\dfrac{x}{R}$ to get the horizontal component, the elementary pressure parallel to the axis of x is

$$\frac{2Px^3\,dx}{R^2 + x^2} \cdot \frac{dy}{\sqrt{a^2 - y^2}}.$$

The limits for y are 0 and a, and integrating with respect to y, and denoting the horizontal pressure by H we have

$$H = 2\pi P. \int_0^R \frac{x^3\,dx}{R^2 + x^2} = 2\pi P. \left\{ \frac{x^2}{2} - \frac{R^2}{2} \log (R^2 + x^2) \right\}_0^R$$

or

$$H = \pi P R^2 . (1 - \log 2). \tag{1}$$

For the x co-ordinate of the point of application of the horizontal force we have to find the integral

$$\int_0^R \frac{x^4\,dx}{R^2 + x^2} = \left(\frac{x^3}{3} - R x^2 + R^3 \arctan \frac{x}{R} \right)_0^R$$

$$= \left(\frac{\pi}{4} - \frac{2}{3} \right) . R^3,$$

and hence

$$\bar{x} = \frac{3\pi - 8}{6(1 - \log 2)} . R. \tag{2}$$

For the corresponding z co-ordinate we have

$$\frac{4P . x^3\,dx}{R^2 + x^2} \int_0^a dy = \frac{4P . x^3 \sqrt{R^2 - x^2} . dx}{R^2 + x^2}$$

and

$$\int \frac{x^3 \sqrt{R^2 - x^2}}{R^2 + x^2} . dx = R^2 . \int \frac{x^3\,dx}{(R^2 + x^2)\sqrt{R^2 - x^2}} - \int \frac{x^5\,dx}{(R^2 + x^2)\sqrt{R^2 - x^2}} .$$

But

$$R^2 . \int_0^R \frac{x^3\,dx}{(R^2 + x^2)\sqrt{R^2 - x^2}} = \left[1 + \frac{1}{\sqrt{2}} . \log (\sqrt{2} - 1) \right] . R^3$$

and

$$\int_0^R \frac{x^5\,dx}{(R^2 + x^2)\sqrt{R^2 - x^2}} = - \left\{ \frac{1}{3} + \frac{1}{\sqrt{2}} . \log (\sqrt{2} - 1) \right\} R^3,$$

so that we have

$$\bar{z} = \frac{4[4 + 3\sqrt{2} . \log(\sqrt{2} - 1)]}{3\pi (1 - \log 2)} . R. \tag{3}$$

In order to get the vertical pressure on the element we have to multiply by $\frac{z}{R}$, and we have for this elementary pressure

$$\frac{2Px^2}{R^2 + x^2} \cdot dx\, dy.$$

The integration with respect to y gives

$$\frac{4P \cdot x^2\sqrt{R^2 - x^2}}{R^2 + x^2} \cdot dx,$$

and hence the vertical pressure is

$$4P \cdot \int_0^R \frac{x^2\sqrt{R^2 - x^2}}{R^2 + x^2} \cdot dx.$$

If V be the vertical pressure the reduction of this integral gives

$$V = \pi P R^2 \cdot (3 - \sqrt{8}). \tag{4}$$

For the x co-ordinate of V we have to find the integral of

$$\frac{2Px^3}{R^2 + x^2}\, dx\, dy.$$

Integrating with respect to y we have

$$4P \cdot \int_0^R \frac{x^3\sqrt{R^2 - x^2}}{R^2 + x^2} \cdot dx.$$

This integral is the same as that found before, and we have

$$\bar{x} = \frac{4\left[4 + 3\sqrt{2} \cdot \log\left(\sqrt{2} - 1\right)\right]}{3\pi(3 - \sqrt{8})} \cdot R. \tag{5}$$

For the z co-ordinate we have the form

$$\frac{2Px^2z}{R^2 + x^2}\, dx\, dy = \frac{2Px^2\, dx}{R^2 + x^2} \cdot \sqrt{a^2 - y^2} \cdot dy.$$

But

$$\int_0^a \sqrt{a^2 - y^2} \cdot dy = \frac{a^2\pi}{4} = (R^2 - x^2) \cdot \frac{\pi}{4};$$

and also

$$\pi P \cdot \int_0^R \frac{x^2(R^2 - x^2)}{R^2 + x^2} \cdot dx = \pi P\left(\frac{5}{3} - \frac{\pi}{2}\right) \cdot R^3.$$

We have therefore

$$z = \frac{10 - 3\pi}{6(3 - \sqrt{8})} \cdot R. \qquad (6)$$

Collecting results and expressing the coefficients by common logarithms we have

$$H = [9.98408] \cdot PR^2, \qquad V = [9.73160] \cdot PR^2,$$
$$\bar{x} = [9.88867] \cdot R, \qquad \bar{x} = [9.80943] \cdot R,$$
$$\bar{z} = [9.55695] \cdot R, \qquad \bar{z} = [9.74724] \cdot R.$$

If we assume $P = 100$, and $R = 22$ feet, we have

$$H = 46658 \text{ pounds}, \qquad V = 26088 \text{ pounds},$$
$$\bar{x} = 17.0 \text{ feet}, \qquad \bar{x} = 14.2 \text{ feet},$$
$$\bar{z} = 7.9 \text{ feet}, \qquad \bar{z} = 12.3 \text{ feet}.$$

[*A. Hall.*]

[The formula quoted in the question is empirical and was derived by Duchemin from experimental results given by Vince, (*Philosophical Transactions of the Royal Society of London*, 1778); by Hutton, (*Resistance of the Air to Bodies in Motion*, Tract 36, 1788); and by Thibault, (*Recherches experimentales sur la resistance de l'air*). Hutton represented the results of his own experiments alone quite exactly by the formula

$$(\cos i)^{1.842 \sin i}.$$

This formula represents the whole group of data however less well than Duchemin's.—Ed.]

EXERCISES.

7

In the theory of perturbations, if the differential equations have the form

$$\frac{d^2\xi}{dt^2} + \frac{k^2(1+m)}{r^3} \cdot \xi = A,$$

prove that

$$k\sqrt{p_0} \cdot \xi = \xi_0 + \xi_1 + \xi_2 + \ldots,$$

where

$$\xi_0 = y_0 \int A x_0 \, dt - x_0 \int A y_0 \, dt + c_1 y_0 - c_2 x_0,$$

$$\xi_{n+1} = y_0 \int B\xi_n x_0\, dt - x_0 \int B\xi_n y_0 dt\,,$$

$$B = \frac{k\sqrt{1+m}}{\sqrt{p_0}} \left(\frac{1}{r_0^{\,3}} - \frac{1}{r^3}\right),$$

in which c_1 and c_2 are constants of integration and p_0, r_0, x_0, y_0 refer to an assumed elliptic orbit. [*Ormond Stone.*]

8

If an ellipse and a rectangular hyperbola have the same centre, and the hyperbola passes through the focus of the ellipse, then at the point of intersection of the curves the ellipse makes equal angles with the hyperbola and the central radius. [*H. A. Newton.*]

9

It is assumed that when a gate in a water pipe is closing the pressure increases uniformly and the discharge decreases uniformly. Investigate an expression for the shortest safe time for closing the gate on the basis of these hypotheses: given the length of the pipe, the velocity of the stream, the working pressure, and the greatest admissible pressure to which the pipe may be exposed. [*W. M. Thornton.*]

10

Required the length of a thread wrapped spirally round the frustum of a given cone, the distance between the spires along the slant height being constant. [*A. B. Nelson.*]

11

In exercise 4, what is the probability that the circle exceeds the average circle? [*Artemas Martin.*]

12

The result

$$-\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + 9r^3}{(r - pq)^2}$$

is given as the equivalent of the function

$$\left(\frac{\beta-\gamma}{\beta+\gamma}\right)^2 + \left(\frac{\gamma-\alpha}{\gamma+\alpha}\right)^2 + \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2,$$

where α, β, γ are the roots of the cubic

$$x^2 + px^2 + qx + r = 0.$$

Is this result correct? [*A. Hall.*]

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Terms of Subscription: $2.00 for six numbers; single numbers 50 cents.

Entered at the Post-office at the University of Virginia as second-class mail matter.